The Transport of Vorticity on Rotating Bernoulli Surfaces of any Geometry in Compressible, Inviscid Flow with Applications to Turbomachine Blade Rows

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The transport of vorticity is analysed for a compressible, inviscid flow which is steady relative to a reference frame rotating at constant angular velocity. It is shown that Helmholtz's vorticity convection theorem does not apply to this flow but nevertheless the vortex lines are transported on a streamsurface which therefore corresponds to the familiar Bernoulli surface.

Explicit, integrated results are obtained for Bernoulli surfaces of any geometry. The transport of the normal component of vorticity is obtained for the general case in closed form, whereas the transport of the streamwise component is closed in form for some cases but involves a time difference integral over the bounding streamlines.

Application is made to a turbomachine blade row where the flow between two consecutive blades is examined. Explicit results are obtained for the streamwise vorticity development in the axial flow configuration in terms of the traverse time integral for a particle, taken around the blade profile. The more general mixed flow configuration is also examined where a closed result is obtained only for the incompressible case.

NOMENCLATURE

Characters

A	cross sectional area
C	constant
C_p	specific heat at constant pressure
$d\mathbf{S}$	elemental surface vector
h	enthalpy
I	rothalpy $[\equiv h + (W^2 - U^2)/2]$
K	compressibility factor (see eq. (2.14))
k	isentropic gas index
l	contour length
M	relative Mach number
$\dot{m}$	mass flow rate
N	contour orthogonal to streamlines
$\mathbf{n}$	unit vector normal to Bernoulli surface
p	pressure
$\mathbf{R}$	location vector
$\mathbf{r}$	radius vector
S	surface
s	entropy
s'	blade pitch
s''	distance along streamline
T	temperature
t	time
t'	Bernoulli sheet thickness
U	blade speed
$\mathbf{V}$	absolute velocity vector
$\mathbf{W}$	relative velocity vector
$\mathbf{z}$	axial co-ordinate
α	fluid deflection on Bernoulli surface
δ	finite small quantity
ρ	density
$\boldsymbol{\Omega}$	rotational speed vector
$\boldsymbol{\omega}$	absolute vorticity vector
$\boldsymbol{\omega}'$	relative vorticity vector

Subscripts

0	absolute stagnation value
1	upstream
2	downstream
A	edge of Bernoulli sheet at $N = 0$
B	edge of Bernoulli sheet at extremity of N
r, θ, z	co-ordinate direction or components (radial, tangential or axial respectively)
or	relative stagnation value (temperature)
n	component in direction normal to Bernoulli surface
N	component normal to stream but tangent to Bernoulli surface
W	component in direction of stream

Superscripts

$\sim$	mean value (vorticity)

Symbols

$\oint$	closed integral
$\times$	vector product
.	scalar product
$(\partial/\partial t)_{\mathbf{R}}$	time derivative at fixed location $\bar{R}$
grad	vector operator (gradient)
div	vector operator (divergence)
curl	vector operator

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INTRODUCTION

Previous work in the field of vorticity convection has been applied predominantly to incompressible flow and relies heavily on Hawthorne's pioneering work (**1**). Horlock and Lakshminarayana (**2**) developed the incompressible case for the inclusion of laminar viscous effects while Marris (**3**) considered application to a rotating frame of reference for incompressible flow with stratified

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density. Smith (**4**) analysed incompressible flow in a rotating passage. Loos (**5**) appears to have considered the effect of compressibility on vorticity convection but he was unable to obtain an explicit expression for the development of the streamwise vorticity component. Previous workers mentioned above also failed to obtain expressions for the development of streamwise vorticity which were explicit in kinematic terms, the awkward vector product of vorticity and velocity being eliminated by substitution of the gradient of the stagnation pressure from the equation of motion. Marsh (**6**) presents an explicit result for a rectilinear, stationary cascade.

All previous work has been left either in general vector terms prior to application to a specified geometry (**7**), (**8**) or has been expressed in terms relating to intrinsic coordinates of the particle path, involving the principal radius of curvature and its normal. Generally, neither of these representations lend themselves to easy visual interpretation in three-dimensional flows, the radius of curvature and the stagnation pressure gradient changing in both magnitude and direction as the particle proceeds.

The present paper derives generalized, explicit results for the development of both normal and streamwise vorticity in compressible flow on Bernoulli-like surfaces which rotate about a fixed axis. The latter feature enables application to be made to rotating turbomachinery blade rows. The turbomachine designer is used to visualizing flow on cylindrical Bernoulli surfaces and to handling the deflection of the flow on such surfaces. Accordingly, the present paper develops the theory in relation to the geometry of a generalized Bernoulli surface and deflections of the flow on that surface which in general has some twisted form. It should be appreciated that the principal radius of curvature of the flow on such a surface does not generally lie in the tangent plane to the surface and may pass from one side of the surface to the other in a complex way. The results presented are explicit in kinematic terms for the incompressible case because both vorticity components are solved for and it transpires that the development of the normal component leads to an integrated result, even for the compressible flow in the rotating reference frame. In the compressible case the expressions while still being intrinsically explicit are left with density terms intact to simplify presentation, though suitable equations of state together with the equation of motion enable results to be obtained in entirely kinematic terms. It is found that in the case of a stationary reference frame the development of the streamwise vorticity in compressible flow can be expressed in an integrated form in terms of particle time lapse.

1 VORTICITY CONVECTION IN COMPRESSIBLE FLOW

Loos (**5**) obtained an expression for the convection of the streamwise vorticity component ω_W in an inviscid, compressible fluid in the absolute (i.e., inertial) reference frame by taking the identity for the triple vector product

$$\mathbf{V} \times (\mathbf{V} \times \boldsymbol{\omega}) \equiv V(\mathbf{V}\omega_v - \boldsymbol{\omega} V) \tag{1.1}$$

where

$$\boldsymbol{\omega} \equiv \text{curl } \mathbf{V} \tag{1.2}$$

and substituting for $(\mathbf{V} \times \boldsymbol{\omega})$ from the equation of motion. The term on the right-hand side in $\boldsymbol{\omega}$ can be eliminated by taking the divergence of (1.1) but the substitution from the equation of motion merely replaces $(\mathbf{V} \times \boldsymbol{\omega})$ by gradients of thermodynamic properties.

Marris (**3**), treating incompressible flow in a rotating reference frame, commenced with the expression

$$(\mathbf{W} \,.\, \text{grad})\left(\frac{\omega'_w}{W}\right) = \frac{2}{WR'}\left(\frac{\mathbf{W}}{W} \times (\mathbf{W} \times \boldsymbol{\omega}') \,.\, \mathbf{n}'\right) - \frac{\mathbf{W}}{W^2} \,.\, \text{curl } (\mathbf{W} \times \boldsymbol{\omega}') \tag{1.3}$$

which is an identity subject to the condition that $\text{div } \mathbf{W} = 0$. This result may be modified for compressible flow by imposing the continuity equation

$$\text{div } \mathbf{W} = -\frac{\mathbf{W}}{\rho} \,.\, \text{grad } \rho \tag{1.4}$$

on its derivation, when it can be shown that

$$(\mathbf{W} \,.\, \text{grad})\left(\frac{\omega'_w}{\rho W}\right) = \frac{1}{\rho}\left\{\frac{2}{WR}\left(\frac{\mathbf{W}}{W} \times (\mathbf{W} \times \boldsymbol{\omega}') \,.\, \mathbf{n}'\right) - \frac{\mathbf{W}}{W^2} \,.\, \text{curl } (\mathbf{W} \times \boldsymbol{\omega}')\right\} \tag{1.5}$$

Equations (1.1) or (1.5) could be used as a starting point for the present work but in order to discuss important features of vorticity convection in compressible flow it is necessary, and involves no more work, to commence with the equation of motion itself, adapted for application to a rotating reference frame and familiar to workers in the turbomachinery field, viz.

$$\left(\frac{\partial \mathbf{W}}{\partial t}\right)_{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{W} + \text{grad } I - T \text{ grad } s = 0 \tag{1.6}$$

Here $\mathbf{R}$ locates a point in the rotating reference frame and $\boldsymbol{\omega}$ is the vorticity in the inertial frame. By using the unsteady continuity eq. (A.1) and taking the curl of eq. (1.6) we obtain a general expression for vorticity convection at eq. (A.3) viz:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho} \,.\, \text{grad } \mathbf{W} + \frac{1}{\rho} \text{grad } T \times \text{grad } s \tag{1.7a}$$

which is applicable to unsteady flow. Because this equation applies to unsteady flow it takes the same form for the rotating reference frame as for the inertial frame. However, the vorticity $\boldsymbol{\omega}$ is always the inertial vorticity while $\mathbf{W}$ is the particle velocity relative to the reference frame under consideration.

For incompressible flow eq. (1.7a) takes the form:

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \boldsymbol{\omega} \,.\, \text{grad } \mathbf{W} \tag{1.7b}$$

which states that the vorticity vector $\boldsymbol{\omega}$ is carried by the fluid particles and is the familiar Helmholtz vorticity convection theorem. In compressible flow eq. (1.7a) shows that neither the vorticity vector $\boldsymbol{\omega}$ nor its modified compressible form $(\boldsymbol{\omega}/\rho)$ is simply carried by the fluid motion when entropy gradients exist in the flow.

In what follows we shall consider flows which are inviscid and adiabatic and therefore isentropic (i.e., $(ds/dt) = 0$) but not homentropic (i.e., grad $s \neq 0$). It follows from the conservation of energy that for such flows $(dI/dt) = 0$. We restrict attention to flows which far upstream are characterized by possessing contours on each of which both I and s are constant, but the constant is different for different contours. By the above, the fluid particles crossing these contours form streamsurfaces on which both I and s will always be constant. In relative steady flow, it follows from eq. (1.6) that the vorticity vectors $\boldsymbol{\omega}$, like the velocity vectors $\mathbf{W}$, are tangent to this surface and in this sense it is like a Bernoulli surface. However, it is not yet demonstrated that the vorticity vectors are transported along these surfaces by the flow.

Let the unit vector normal to the above Bernoulli-like surface be $\mathbf{n}$. The term $\boldsymbol{\omega}/\rho$. grad $\mathbf{W}$ in (1.7a) states that $(\boldsymbol{\omega}/\rho)$ is carried by the fluid and therefore develops no n-component, however the term $1/\rho$ grad $T \times$ grad s states that there is a further change in $(\boldsymbol{\omega}/\rho)$ and therefore $(\boldsymbol{\omega}/\rho)$ is only transported along this Bernoulli-like surface if its n-component (i.e., $\mathbf{n}\,.\,(1/\rho)$ grad $T \times$ grad s) is zero. Now since this is a surface of constant entropy then:

$$\text{grad } s = \mathbf{n}\,\frac{\partial s}{\partial n} \tag{1.8}$$

and it follows that for steady relative flow

$$\mathbf{n}\,.\,\frac{1}{\rho}\,\text{grad } T \times \text{grad } s$$

$$= \mathbf{n}\,.\,\frac{1}{\rho}\,\text{grad } T \times \mathbf{n}\,\frac{\partial s}{\partial n} \equiv 0 \tag{1.9}$$

Hence, although $(\boldsymbol{\omega}/\rho)$ is not simply carried with the fluid particles it is nevertheless transported over this Bernoulli-like surface by the fluid motion.

In the subsequent development it is found convenient to employ a local orthogonal co-ordinate system $\mathbf{n}$–$\mathbf{N}$–$(\mathbf{W}/W)$ forming a right-handed set, where (see Fig. 1)

$\mathbf{W}/W \equiv$ unit vector in the relative stream direction and therefore tangent to the Bernoulli-like surface;
$\mathbf{n} \equiv$ unit vector normal to the Bernoulli-like surface;
$\mathbf{N} \equiv$ unit vector tangent to the Bernoulli-like surface but normal to $\mathbf{W}$.

From these definitions it follows that:

$$\mathbf{n} \times \mathbf{N} = \frac{\mathbf{W}}{W} \tag{1.10a}$$

$$\mathbf{N} \times \frac{\mathbf{W}}{W} = \mathbf{n} \tag{1.10b}$$

$$\frac{\mathbf{W}}{W} \times \mathbf{n} = \mathbf{N} \tag{1.10c}$$

Now that the similarity of the above surface to a Bernoulli surface has been established, it is henceforward referred to by this appellation.

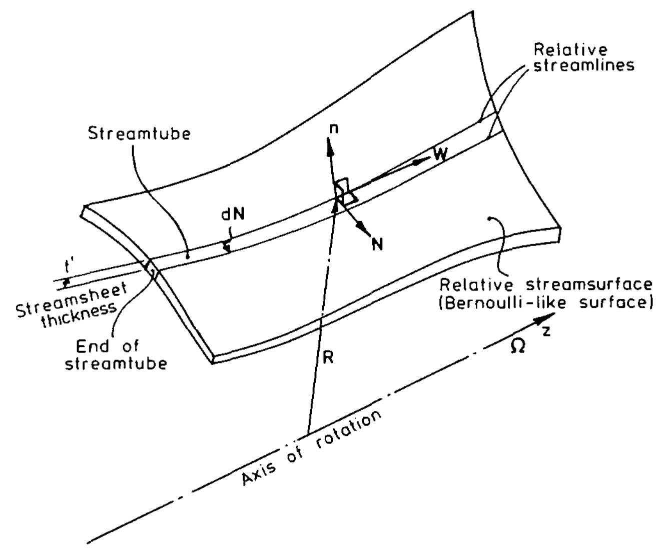

Fig. 1. Illustrating features of an arbitrary relative streamsheet and the local co-ordinate system

2 CONVECTION OF THE VORTICITY COMPONENTS

We may resolve the vorticity vector $\boldsymbol{\omega}$ in any direction indicated by a unit vector $\mathbf{b}$, thus

$$\omega_b = \boldsymbol{\omega}\,.\,\mathbf{b} \tag{2.1}$$

hence

$$\frac{d}{dt}\left(\frac{\omega_b}{\rho}\right) = \frac{d}{dt}\left(\frac{\boldsymbol{\omega}\,.\,\mathbf{b}}{\rho}\right) = \frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right).\,\mathbf{b} + \frac{\boldsymbol{\omega}}{\rho}\,.\,\frac{d\mathbf{b}}{dt} \tag{2.2}$$

when by eq. (1.7a)

$$\frac{d}{dt}\left(\frac{\omega_b}{\rho}\right) = \mathbf{b}\,.\left(\frac{\boldsymbol{\omega}}{\rho}\,.\,\text{grad}\right)\mathbf{W} + \frac{\boldsymbol{\omega}}{\rho}\,.\,\frac{d\mathbf{b}}{dt} + \frac{\mathbf{b}}{\rho}\,.\,\text{grad } T \times \text{grad } s \tag{2.3}$$

It has been established that ω_n is zero, hence we may express $\boldsymbol{\omega}$ in terms of its streamwise (ω_W) and normal (ω_N) components only (see Fig. 1).

Consider first the normal component, then $\mathbf{b} = \mathbf{N}$ and as derived at (A.27), eq. (2.2) may be written:

$$\frac{d}{dt}\left(\omega_N W t' + C_1\,\frac{W^2 - U^2}{2}\right) = 0 \tag{2.4}$$

where t' is the normal thickness of the Bernoulli sheet. This may be integrated for the particle, along the streamline, to yield the following result which agrees with Marsh (**6**) for a non-rotating, rectilinear cascade of blades in a Bernoulli sheet of constant thickness

$$\omega_N = \frac{C_2}{Wt'} - C_1\,\frac{W^2 - U^2}{2Wt'} \tag{2.5}$$

where

$$C_1 \equiv \frac{t'_1}{C_p}\left(\frac{\partial s}{\partial n}\right)_1 \quad \text{and is dimensionless} \tag{2.6}$$

while

$$C_2 = \omega_{N_1} W_1 t'_1 + C_1\,\frac{W_1^2 - U_1^2}{2} \tag{2.7a}$$

by eq. (1.6) and eq. (2.6)

$$= \frac{\omega_{N_1} W_1 t'_1}{C_p T_1} \left\{ I_1 + \frac{1}{\omega_{N_1} W_1} \left(\frac{\partial I}{\partial n} \right)_1 \right\} \tag{2.7b}$$

where subscript 1 designates any arbitrary reference point on the Bernoulli surface.

Considering now the vorticity component resolved in the streamwise direction, then $\mathbf{b} = (\mathbf{W}/W)$ and as derived at (A.42), eq. (2.2) may be written:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho W}\right) = -\frac{2C_2 + C_1 U^2}{\rho W^2 t'}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \mathbf{\Omega} \cdot \mathbf{n}\right) - \frac{C_1}{\rho t'}\left(\mathbf{\Omega} \cdot \mathbf{n} + \frac{\Omega^2}{W} \mathbf{N} \cdot \mathbf{r}\right) \tag{2.8}$$

where $\mathrm{d}\alpha$ is the change in stream direction measured *on* the relative streamsurface (i.e., about $\mathbf{n}$), the sign being taken from eq. (A.14) viz:

$$\mathrm{d}\left(\frac{\mathbf{W}}{W}\right)_N = -\mathbf{N}\,\mathrm{d}\alpha \tag{2.9}$$

Unlike the equation for the normal vorticity the streamwise vorticity eq. (2.8) is not in the general case integrable explicitly in terms of the angle α. The alternative form in terms of the particle traverse time t given at eq. (A.51) viz:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho W}\right) = -\frac{2C_2 + C_1 U^2}{2\,\mathrm{d}\dot{m}} \frac{\partial(\mathrm{d}t)}{\partial N} \frac{\mathrm{d}N}{\mathrm{d}t} - \frac{C_1}{\rho t'}\left(\mathbf{\Omega} \cdot \mathbf{n} + \frac{\Omega^2}{W} \mathbf{N} \cdot \mathbf{r}\right) \tag{2.10}$$

can be integrated along the flow for a rotating cylindrical Bernoulli surface, when U = constant and $\mathbf{\Omega} \cdot \mathbf{n} = 0 = \mathbf{N} \cdot \mathbf{r}$, or for any Bernoulli surface in the non-rotating case when $\mathbf{\Omega} = 0 = U$. In eq. (2.10), from eq. (A.50),

$$\mathrm{d}\dot{m} = \rho W t'\,\mathrm{d}N = \text{constant} \tag{2.11}$$

is the mass flow rate along a streamtube of normal width $\mathrm{d}N$ on the Bernoulli surface. t is the time for a particle to traverse a distance along the streamtube.

Since the two integrable cases above reduce to the same general form they can be developed as a common case, when eq. (2.10) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho W}\right) = C_3 \frac{\partial(\mathrm{d}t)}{\partial N} \frac{\mathrm{d}N}{\mathrm{d}t}$$

which upon integration along the streamtube yields

$$\left[\frac{\omega_W}{\rho W}\right]_1^2 = C_3 \frac{\partial t}{\partial N}\,\mathrm{d}N \tag{2.12}$$

where $(\partial t/\partial N) \cdot \mathrm{d}N$ is the time difference that particles on adjacent streamlines separated by normal distance $\mathrm{d}N$ take to traverse between normals N_1 and N_2.

In this case from eq. (2.7)

$$C_2 + \frac{C_1 U^2}{2} = \omega_{N_1} W_1 t'_1 K \tag{2.13}$$

where

$$K = \frac{1}{C_p T_1}\left[I_1 - \frac{U_1^2}{2} + \frac{1}{\omega_{N_1} W_1}\left(\frac{\partial I}{\partial n}\right)_1\right] \tag{2.14}$$

For a constant energy flow field grad $I = 0$, then

$$\left(\frac{\partial I}{\partial n}\right)_1 = 0 \tag{2.15}$$

when for a gas

$$K = 1 + \frac{W_1^2}{2C_p T_1} = \frac{T_{or}}{T_1} = 1 + \frac{k-1}{2} M^2 \tag{2.16}$$

where M is the relative Mach number and T_{or} is the relative stagnation temperature.

If we consider a Bernoulli surface which has uniform upstream flow so that all quantities are constant along N_1 (see Fig. 2), the upstream normal, then by reference to eqs. (2.10) and (2.11), (2.12) may be written

$$[\omega_W t'\,\mathrm{d}N]_1^2 = (\text{constant}) \frac{\partial(t)}{\partial N}\,\mathrm{d}N \tag{2.17}$$

where by eq. (2.13)

$$\text{constant} = -\left(C_2 + C_1 \frac{U^2}{2}\right) = -\omega_{N_1} W_1 t'_1 K$$

is now constant along streamlines and across the upstream flow and is therefore a unique constant for the whole Bernoulli surface.

Equation (2.17) may now be integrated across the flow from A to B along the normals to give:

$$\int_A^B (\omega_W t'\,\mathrm{d}N)_2 = (\omega_W t' N)_1 - \omega_{N_1} W_1 t'_1 K (t_B - t_A) \tag{2.18}$$

$$\bar{\omega}_{W_2} = \omega_{W_1} \frac{t'_1 N_1}{A_2} - \frac{\omega_{N_1} W_1 t'_1 K}{A_2}(t_B - t_A) \tag{2.19}$$

where $\bar{\omega}_{W_2}$ is the sheet cross-sectional area mean value of ω_{W_2} on N_2 defined by

$$\bar{\omega}_{W_2} \equiv \frac{\int_A^B \omega_{W_2} t'_2\,\mathrm{d}N_2}{\int_A^B t'_2\,\mathrm{d}N_2} = \frac{\int \omega_{W_2}\,\mathrm{d}A_2}{A_2} \tag{2.20}$$

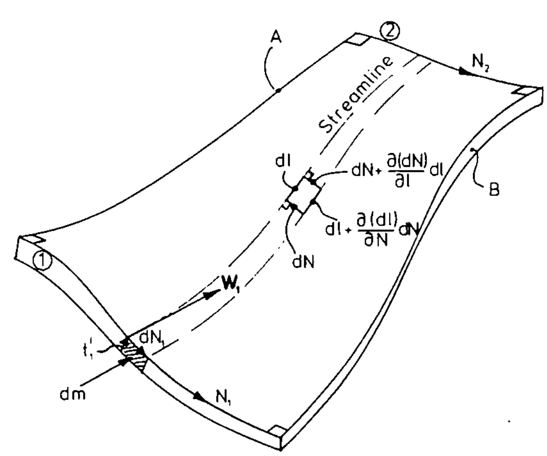

Fig. 2. Illustrating features of a relative streamtube

then by continuity

$$\rho_1 W_1 A_1 = \rho_1 W_1 t_1' N_1 = \rho_2 W_2 A_2 \qquad (2.21)$$

eq. (2.19) may be written

$$\tilde{\omega}_{W_2} = \frac{\rho_2 W_2}{\rho_1 W_1}\left(\omega_{W_2} - \frac{\omega_{N_1} W_1 K}{N_1}(t_B - t_A)\right) \qquad (2.22)$$

This result is of special interest because it demonstrates that the conversion of normal vorticity into streamwise vorticity in compressible flow depends on the difference of time taken for particles to travel between normals along the bounding streamlines A and B. A similar conclusion can be deduced in incompressible flow by applying the Kelvin theorem which states that the vorticity vector is carried by the fluid particles. However, as discussed earlier, in compressible flow the vorticity vector is not so carried.

If the flow detail on the Bernoulli surface has been determined by some suitable method **(9)**-**(11)** eqs. (2.5) and (2.8) or (2.10) (or where applicable (2.19)) may be used to determine the vorticity distribution. Usually it is the relative vorticity $\boldsymbol{\omega}'$ that is required. This may be recovered from the relationship:

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + 2\boldsymbol{\Omega} \qquad (2.23)$$

when resolving components gives:

$$\omega_N' = \omega_N - 2\mathbf{N}\,.\,\boldsymbol{\Omega} = \omega_N - 2\Omega N_z \qquad (2.24a)$$

$$\omega_W' = \omega_W - 2\frac{\mathbf{W}}{W}\,.\,\boldsymbol{\Omega} = \omega_W - 2\Omega\frac{W_z}{W} \qquad (2.24b)$$

3 SMALL PERTURBATION ANALYSIS

Consider small perturbations of a rotating cylindrical Bernoulli surface then $\mathbf{n} = (\mathbf{r}/r)$ and

$$\boldsymbol{\Omega}\,.\,\mathbf{n} = 0 = \mathbf{N}\,.\,\mathbf{r} \qquad (3.1)$$

or for a general stationary Bernoulli surface then eqs. (2.5) and (2.8) yield

$$\frac{\delta\omega_N}{\omega_N} = -\left(\frac{C_1 W}{\omega_N t'} + 1\right)\frac{\delta W}{W} - \frac{\delta t'}{t'} + \left(\frac{U}{W}\right)^2 \frac{C_1 W}{\omega_N t'}\cdot\frac{\delta r}{r} \qquad (3.2)$$

$$\frac{\delta\omega_W}{\omega_W} = \frac{\delta W}{W} + \frac{\delta\rho}{\rho} - 2\left(\frac{\omega_N}{\omega_W} + \frac{C_1 W}{2\omega_W t'}\right)\delta\alpha \qquad (3.3)$$

It can be seen from eq. (3.2) that for compressible flow ($C_1 \neq 0$) on a rotating Bernoulli surface ($U \neq 0$), the surface distortion has a first order effect on the development of the normal vorticity as evidenced by the term in $\delta r/r$. This effect is absent on a stationary Bernoulli surface and is always absent from the streamwise vorticity development, eq. (3.3).

Using the continuity equation

$$\frac{\delta W}{W} + \frac{\delta\rho}{\rho} + \frac{\delta A}{A} = 0 \qquad (3.4)$$

for a streamtube, where A is the streamtube normal section area, eqs. (3.2) and (3.3) may be written

$$\delta\omega_N = \left|-\left(1 + \frac{C_1 W}{\omega_N t'}\right)\frac{\delta W}{W} - \frac{\delta t'}{t'} + \left(\frac{U}{W}\right)^2 \frac{C_1 W}{\omega_N t'}\frac{\delta r}{r}\right|\cdot\omega_N \qquad (3.5)$$

$$\delta\omega_W = -\frac{\delta A}{A}\omega_W - 2\omega_N\left(1 + \frac{C_1 W}{2\omega_N t'}\right)\delta\alpha \qquad (3.6)$$

Equation (3.5) demonstrates that the normal vorticity undergoes simple amplification with no interaction from the streamwise vorticity. This amplification occurs as a consequence of velocity diffusion ($\delta W < 0$), decreasing Bernoulli sheet thickness ($\delta t' < 0$) or, if rotation is present ($U \neq 0$), with increase in radius ($\delta r > 0$) for compressible flow ($C_1 > 0$). The effect of compressibility ($C_1 > 0$) is to increase the velocity diffusion effect. It can be seen from eq. (2.16) that for constant energy flows (grad $I = 0$) the term

$$\left(1 + \frac{C_1 W}{2t'\omega_N}\right) = \left(\frac{T_{or}}{T}\right) = 1 + \frac{k-1}{2}M^2 \qquad (3.7)$$

when

$$\frac{C_1 W}{t'\omega_N} = (k-1)M^2 \qquad (3.8)$$

so that this factor is anticipated to be positive in sign. It is interesting to note that surface deflection itself has no effect. It will be seen from eq. (3.5) that Loos' **(5)** assumption of the constancy of ω_N when $\delta t' = 0 = U$ is not correct, except for a flow with zero diffusion ($\delta W = 0$) which is not a boundary layer situation.

Equation (3.6) demonstrates that the streamwise vorticity is amplified by a decreasing streamtube area while normal vorticity is converted to streamwise vorticity by the flow deflection on the Bernoulli surface. Equation (3.5) suggests that this conversion is increased by compressibility. Loos **(5)** demonstrated this conversion effect for stationary Bernoulli surfaces in a thin boundary layer, but the above development shows that the result is applicable to rotating Bernoulli surfaces and is independent of his assumptions of the constancy of ω_N and $(\partial T/\partial n)$ or that $(\partial p/\partial n) = 0$ which would only apply to plane Bernoulli surfaces. It is clear that Loos' **(5)** results are applicable to boundary layer development on a turbomachine rotor hub.

For the special case of incompressible flow ($C_1 = 0$) with zero streamwise entry vorticity $\omega_W = 0$, eq. (3.6) reduces to the familiar result of Squire and Winter **(13)** viz

$$\delta\omega_W = -2\omega_N\,\delta\alpha \qquad (3.9)$$

which is in fact independent of the constancy of W, for small deflections.

4 APPLICATION TO TURBOMACHINE BLADE CASCADES

Consider an annular cascade of blades producing a finite flow deflection on Bernoulli surfaces which are distorted only slightly from circular cylinders. In this configuration (Fig. 3) the surface normal direction $\mathbf{n}$ has been taken as radially outwards so that positive deflection ($d\alpha$) in the figure, is towards the meridional plane.

The conditions

$$\boldsymbol{\Omega}\,.\,\mathbf{n} = \mathbf{N}\,.\,\mathbf{r} = dU = 0 \qquad (4.1)$$

apply to this surface so that eq. (2.22) is applicable to the flow. The upstream and downstream normals, N_1 and N_2 respectively, are shown in the figure and the

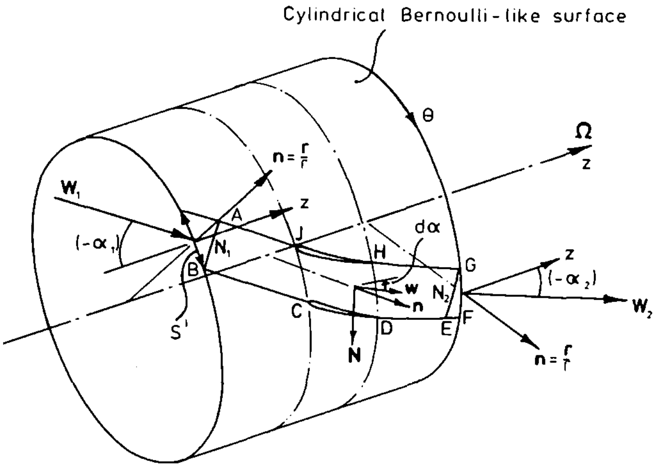

Fig. 3. Details of flow through consecutive blades on a circular cylindrical streamsurface

traverse times t_A and t_B of eq. (2.22) become the times to traverse paths AG and BE respectively. Upstream and downstream of the blades the flow is considered periodic over the angular blade pitch so that

$$t_{KJ} = t_{BC} \tag{4.2a}$$

$$t_{HG} = t_{DF} \tag{4.2b}$$

now

$$\begin{aligned} t_{BE} - t_{AG} &= (t_{BF} - t_{EF}) - (t_{KG} - t_{KA}) \\ &= (t_{BC} + t_{CD} + t_{DF} - t_{EF}) \\ &\quad - (t_{KJ} + t_{JH} + t_{HG} - t_{KA}) \end{aligned}$$

by periodicity

$$= -(t_{JH} - t_{CD}) - t_{EF} + t_{KA} \tag{4.3}$$

but in the downstream and upstream uniform flow

$$t_{EF} = \frac{-s' \sin\alpha_2}{W_2} \tag{4.4}$$

and

$$t_{KA} = \frac{-s' \sin\alpha_1}{W_1} \tag{4.5}$$

while

$$t_{JH} - t_{CD} = \oint_{\text{blade}} dt \tag{4.6}$$

is taken around the blade in the sense of positive $\mathbf{n}$ and is in sign agreement with Hawthorne (7). Substituting eqs. (4.3), (4.4), (4.5), (4.6) in eq. (2.22) we obtain

$$\tilde{\omega}_{W_2} = \frac{\rho_2 W_2}{\rho_1 W_1}\left[\omega_{W_1} - \frac{\omega_{N_1} W_1 K}{s' \cos\alpha_1} \times \left\{s'\left(\frac{\sin\alpha_2}{W_2} - \frac{\sin\alpha_1}{W_1}\right) - \oint_{\text{blade}} dt\right\}\right] \tag{4.7}$$

where K is defined at eq. (2.14) and

$$N_1 = s' \cos\alpha_1 \tag{4.8}$$

$$N_2 = s' \cos\alpha_2 \tag{4.9}$$

so that by eq. (2.20)

$$\tilde{\omega}_{W_2} = \frac{1}{s' \cos\alpha_2}\int \omega_{W_2} \cos\alpha_2 \, ds' = \frac{1}{s'}\int \omega_{W_2} \, ds' \tag{4.10}$$

is the pitch averaged downstream streamwise vorticity for constant t'_2. It should be observed that the angles α (Fig. 3) measured to the axis of rotation are negative in order to comply with the condition that $\mathbf{n} = (\mathbf{r}/r)$. The result, eq. (4.7), does not assume constant sheet thickness nor thin or closely spaced blades and is valid for a stator blade row independently of the twisting of the Bernoulli surface. However, with these restrictions applied, the results agree with Marsh (**6**).

For incompressible flow with constant sheet thickness

$$K = 1, \quad \rho_1 = \rho_2 \quad \text{and} \quad \frac{W_2}{W_1} = \frac{\cos\alpha_1}{\cos\alpha_2}$$

so that eq. (4.7) reduces to

$$\tilde{\omega}_{W_2} = \omega_{W_1}\frac{\cos\alpha_1}{\cos\alpha_2} - \frac{\omega_{N_1}}{\cos\alpha_1 \cos\alpha_2} \times \left\{\tfrac{1}{2}(\sin 2\alpha_2 - \sin 2\alpha_1) - \frac{W_Z}{s'}\oint dt\right\} \tag{4.11}$$

which agrees with Marsh and Came (**12**) and with Loos (**14**) and Hawthorne (7) when $\omega_{W_1} = 0$, after observing the sign convention for α. However, Hawthorne's result applied only to closely spaced blades and while Marsh and Came follow an analysis which is applicable to blades of finite spacing, they do not define the mean value $\tilde{\omega}_{W_2}$ so obtained. Both the above analyses were applied to a plane Bernoulli surface but eq. (4.11) is applicable to a Bernoulli surface of any twist for a stator blade row.

If one considers closely spaced blades of zero thickness under conditions of incompressibility and constant sheet thickness then the axial velocity W_Z is constant while

$$dN = ds' \cos\alpha \tag{4.12}$$

where ds' is the blade pitch and

$$W_Z = W \cos\alpha \tag{4.13}$$

so that the relationship between the time difference and the deflection at eq. (A.48) may be integrated along the flow thus:

$$\begin{aligned} \int_1^2 \frac{\partial(dt)}{\partial N} dN &= \left(\frac{ds'}{W_Z}\right)\int_1^2 2\cos^2\alpha \, d\alpha \\ &= \frac{ds'}{W_Z}\left(\tfrac{1}{2}\sin 2\alpha + \alpha\right)_1^2 = t_2 - t_1 \end{aligned} \tag{4.14}$$

where 1 is an upstream station and 2 a downstream station, but also by eqs. (4.3), (4.4), (4.5) and (4.6).

$$\begin{aligned} t_2 - t_1 &= -ds' \sin\alpha_1 \frac{\cos\alpha_1}{W_Z} \\ &\quad + ds' \sin\alpha_2 \frac{\cos\alpha_2}{W_Z} - \oint_{\text{blade}} dt \\ &= \frac{ds'}{W_Z}\left\{\tfrac{1}{2}(\sin 2\alpha_2 - \sin 2\alpha_1)\right\} - \oint_{\text{blade}} dt \end{aligned} \tag{4.15}$$

therefore

$$\int_{\text{blade}} dt = -\frac{ds'}{W_Z}(\alpha_2 - \alpha_1) \qquad (4.16)$$

as shown by Came and Marsh (**12**).

The direct effects of compressibility may be seen from eq. (4.7) where for given velocity vectors $\mathbf{W}_1$ and $\mathbf{W}_2$ the integral

$$\left(-\int_{\text{blade}} dt\right)$$

which has the same sign as

$$s\left(\frac{\sin \alpha_2}{W_2} - \frac{\sin \alpha_1}{W_1}\right)$$

will increase together with K (see eq. (2.16)). These combine to increase the rate of conversion of normal into streamwise vorticity. The effect of the density ratio (ρ_2/ρ_1) across the cascade is one of simple scaling of the upstream vorticity vector, the scaling factor being greater than unity for a compressor cascade and less than unity for a turbine cascade.

For incompressible flow ($C_1 = 0$, $K = 1$, $\rho_1 = \rho_2$) it will be observed that all terms relating to rotation in eq. (2.10) vanish and eq. (2.22) becomes:

$$\tilde{\omega}_{W_2} = \frac{W_2}{W_1}\left\{\omega_{W_1} - \frac{\omega_{N_1} W_1}{N_1}(t_B - t_A)\right\} \qquad (4.17)$$

which applies to a rotating Bernoulli surface of any geometry which has uniform upstream conditions. If this is applied to perturbations of an axi-symmetric (Fig. 4) surface of the general turbomachine, then following the earlier part of this section but noting that the angles are measured to the meridional plane and that in eqs. (4.4) and (4.5)

$$s_1' = r_1\theta \qquad (4.18)$$

and

$$s_2' = r_2\theta \qquad (4.19)$$

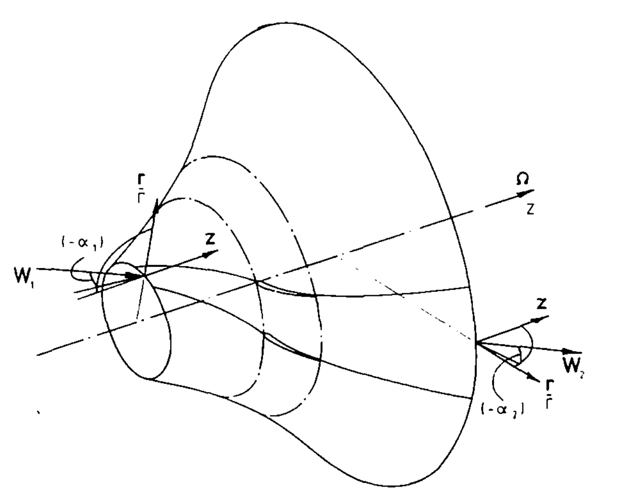

Fig. 4. Details of flow through consecutive blades on an arbitrary streamsurface of revolution

where θ is the angular blade pitch, then eq. (4.7) becomes

$$\tilde{\omega}_{W_2} = \frac{W_2}{W_1}\left[\omega_{W_1} - \frac{\omega_{N_1} W_1}{r_1\theta \cos \alpha_1} \times \left\{\theta\left(\frac{r_2 \sin \alpha_2}{W_2} - \frac{r_1 \sin \alpha_1}{W_1}\right) - \oint_{\text{blade}} dt\right\}\right] \qquad (4.20)$$

The flow in the vaneless annular diffuser of a centrifugal compressor, subsequent to leaving the impeller blades, may be analysed using eq. (4.20) with the time integral put equal to zero.

The principal difficulty in applying eqs. (4.7) or (4.20) lies in the determination of the time integral which demands a knowledge of the blade surface velocity distribution. Even with this knowledge, the evaluation of the integral is difficult if a stagnation point exists, such as will be the case for rounded leading edges or non-zero incidence if the blade has a cusped leading edge, which includes the case of zero thickness blades. As will be seen from eq. (A.44) a stagnation point leads to a singularity in the integrand. Analysis of plane incompressible flow around a circular cylinder shows that in fact particles approach but do not reach the stagnation points, so that in fact the time integral for flow over the surface from the forward to the rear stagnation point is infinite and the integral as well as the integrand is infinite. Of course it does not follow that the integral taken around the entire surface is infinite and other considerations suggest that this is not so.

CONCLUSIONS

It has been shown that in compressible flow, Helmholtz's vorticity convection theorem does not apply but that nevertheless the vortex lines are transported by the fluid motion over a streamsurface which may therefore be regarded as a Bernoulli surface, because it has constant rothalpy.

Equations for the development of the vorticity components on such a Bernoulli surface, of any geometry, which rotates about a fixed axis of rotation and is in a steady state in the rotating reference frame have been obtained for a particle in inviscid, compressible flow. The equation for the normal component of vorticity can be obtained in an explicit, integrated form, eq. (2.5), in all cases. The equation for the streamwise vorticity component of a particle in terms of the fluid deflection α eq. (2.8), although explicit is not generally integrable but can be integrated for the case of incompressible flow on a rotating circular cylindrical Bernoulli surface with constant axial velocity (eqs. (4.11) and (4.14)). The streamwise vorticity component in terms of the difference in time for particles to traverse the bounding streamlines of the Bernoulli surface can be integrated for the compressible flow for rotating circular cylindrical Bernoulli surfaces eq. (4.7). The case of compressible flow on a rotating radial Bernoulli sheet of constant thickness takes an explicit integral form if an equation of state relating density to velocity is used (e.g., a perfect gas) but is not developed here.

For stationary Bernoulli surfaces of any geometry the compressible flow result is obtainable in an integrated form in terms of the time difference previously mentioned eq. (2.22).

For incompressible flow an integrated result is available for rotating Bernoulli surfaces of any geometry in terms of the time difference (eq. (2.22) with $\rho_1 = \rho_2$ and $K = 1$).

A small perturbation analysis can be carried out which relates the basic flow features to the vorticity development.

Application is made to turbomachine blade cascades which are represented by periodic flows around a single blade, on axi-symmetric Bernoulli surfaces. In this situation the time difference for particles to traverse the bounding streamlines can be reduced, because of the periodicity condition, to the integral of the traverse time taken around the blade profile. A general result for incompressible flow is given at eq. (4.20) and a compressible flow result for the case when the Bernoulli surface is a circular cylinder at eq. (4.7), the specific incompressible result for this, which is well known, is given at eq. (4.11).

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APPENDIX

Taking the curl of the equation of motion

$$\text{curl}\left(\left(\frac{\partial \mathbf{W}}{\partial t}\right)_{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{W} + \text{grad } I - T \text{ grad } s\right) = 0$$

$$\frac{\partial}{\partial t}(\text{curl } \mathbf{W})_{\mathbf{R}} + \text{curl } (\boldsymbol{\omega} \times \mathbf{W}) + 0 - \text{curl } (T \text{ grad } s) = 0$$

$$\left(\frac{\partial \boldsymbol{\omega}'}{\partial t}\right)_{\mathbf{R}} - 2\left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)_{\mathbf{R}} + \mathbf{W} \cdot \text{grad } \boldsymbol{\omega} + \boldsymbol{\omega} \text{ div } \mathbf{W} - \boldsymbol{\omega} \cdot \text{grad } \mathbf{W} - \mathbf{W} \text{ div } \boldsymbol{\omega} - T \text{ curl grad } s - \text{grad } T \times \text{grad } s = 0$$

that is,

$$\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\right)_{\mathbf{R}} + \mathbf{W} \cdot \text{grad } \boldsymbol{\omega} + \boldsymbol{\omega} \text{ div } \mathbf{W} - \boldsymbol{\omega} \cdot \text{grad } \mathbf{W} - 0 - 0 - \text{grad } T \times \text{grad } s = 0$$

since curl grad $\equiv 0$ div $\boldsymbol{\omega}$ = div curl $\mathbf{V} \equiv 0$ also

$$\left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)_{\mathbf{R}} = 0$$

Now by continuity

$$\frac{1}{\rho}\frac{d\rho}{dt} + \text{div } \mathbf{W} = 0 \qquad \text{(A.1)}$$

and particle convection

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t_{\mathbf{R}}} + \mathbf{W} \cdot \text{grad } (\ ) \qquad \text{(A.2)}$$

the above may be written

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho} \cdot \text{grad } \mathbf{W} + \frac{1}{\rho} \text{grad } T \times \text{grad } s \qquad \text{(A.3)}$$

When $\mathbf{b} = \mathbf{N}$ then from eq. (13)

$$\frac{d}{dt}\left(\frac{\omega_{\mathbf{N}}}{\rho}\right) = \mathbf{N} \cdot \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \text{grad}\right)\mathbf{W} + \frac{\boldsymbol{\omega}}{\rho} \cdot \frac{d\mathbf{N}}{dt} + \frac{\mathbf{N}}{\rho} \cdot \text{grad } T \times \text{grad } s \qquad \text{(A.4)}$$

but

$$(\boldsymbol{\omega} \cdot \text{grad})\mathbf{W} = \left\{\left(\omega_{\mathbf{W}} \frac{\mathbf{W}}{W} + \omega_{\mathbf{N}}\mathbf{N}\right) \cdot \text{grad}\right\}\mathbf{W} = \frac{\omega_{\mathbf{W}}}{W}(\mathbf{W} \cdot \text{grad})\mathbf{W} + \omega_{\mathbf{N}}\mathbf{N} \cdot \text{grad } \mathbf{W} = \frac{\omega_{\mathbf{W}}}{W}\frac{d\mathbf{W}}{dt} + \omega_{\mathbf{N}}\mathbf{N} \cdot \text{grad } \mathbf{W} \qquad \text{(A.5)}$$

and since

$$\mathbf{N} \cdot \mathbf{W} = 0 \qquad \text{(A.6)}$$

then

$$0 = \text{grad } (\mathbf{N} \cdot \mathbf{W}) \equiv \mathbf{N} \cdot \text{grad } \mathbf{W} + \mathbf{W} \cdot \text{grad } \mathbf{N} + \mathbf{N} \times \text{curl } \mathbf{W} + \mathbf{W} \times \text{curl } \mathbf{N}$$

therefore in steady relative flow

$$\mathbf{N} \cdot \text{grad } \mathbf{W} = -\frac{d\mathbf{N}}{dt} - \mathbf{N} \times \text{curl } \mathbf{W} - \mathbf{W} \times \text{curl } \mathbf{N}$$

and

$$\mathbf{N} \cdot (\mathbf{N} \cdot \text{grad})\mathbf{W} = -\mathbf{N} \cdot \frac{d\mathbf{N}}{dt} - \mathbf{N} \cdot \mathbf{N} \times \text{curl } \mathbf{W}$$
$$- \mathbf{N} \cdot \mathbf{W} \times \text{curl } \mathbf{N}$$
$$= 0 - 0 - \mathbf{N} \times \mathbf{W} \cdot \text{curl } \mathbf{N}$$

(*Note:* $d\mathbf{N}/dt$ must be normal to $\mathbf{N}$ since $\mathbf{N}$ is unit vector) by eq. (8).

$$= -W\mathbf{n} \cdot \text{curl } \mathbf{N} \tag{A.7}$$

By continuity for the streamsheet of normal thickness

$$\mathbf{t}' = t'\mathbf{n} \tag{A.8}$$

then

$$0 = \oint_l \rho \mathbf{W} \cdot \mathbf{t}' \times d\mathbf{l} = \oint_l \rho t' \mathbf{W} \times \mathbf{n} \cdot d\mathbf{l} \tag{A.9}$$

where $d\mathbf{l}$ is an element of contour drawn on the streamsurface, hence by definition of curl

$$\mathbf{n} \cdot \text{curl } (\rho t' \mathbf{W} \times \mathbf{n}) \equiv \underset{l \to 0}{Lt} \oint_l \rho t' \mathbf{W} \times \mathbf{n} \cdot d\mathbf{l} \tag{A.10}$$

by eqs. (9) and (8) then

$$\mathbf{n} \cdot \text{curl } (\rho t' W \mathbf{N}) = 0$$
$$\mathbf{n} \cdot (\rho t' W \text{ curl } \mathbf{N} + \text{grad } (\rho t' W) \times \mathbf{N}) = 0$$
$$\rho t' W \mathbf{n} \cdot \text{curl } \mathbf{N} + \text{grad } (\rho t' W) \cdot \mathbf{N} \times \mathbf{n} = 0$$
$$\rho t' W \mathbf{n} \cdot \text{curl } \mathbf{N} - \frac{\mathbf{W}}{W} \cdot \text{grad } (\rho t' W) = 0$$

therefore

$$W\mathbf{n} \cdot \text{curl } \mathbf{N} = \frac{\mathbf{W}}{\rho t' W} \cdot \text{grad } (\rho t' W) \tag{A.11}$$

Now by eqs. (7) and (11) for steady relative flow

$$\mathbf{N} \cdot (\mathbf{N} \cdot \text{grad})\mathbf{W} = -\frac{1}{\rho t' W} \frac{d}{dt} (\rho t' W) \tag{A.12}$$

but

$$\frac{d}{dt}\left(\frac{\mathbf{W}}{W}\right) = \frac{1}{W} \frac{d\mathbf{W}}{dt} - \frac{\mathbf{W}}{W^2} \frac{d\mathbf{W}}{dt}$$

therefore

$$\frac{1}{W} \frac{d\mathbf{W}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{W}}{W}\right) + \frac{\mathbf{W}}{W^2} \frac{dW}{dt} \tag{A.13}$$

while

$$d\left(\frac{\mathbf{W}}{W}\right)_{\mathbf{N}} = -\mathbf{N}\, d\alpha \tag{A.14}$$

where $d\alpha$ is angular deflection of the stream vector $\mathbf{W}/W$ measured on the streamsurface (i.e., about $\mathbf{n}$), hence

$$\frac{\mathbf{N}}{W} \cdot \frac{d\mathbf{W}}{dt} = \mathbf{N} \cdot \left(\frac{d}{dt}\left(\frac{\mathbf{W}}{W}\right) + \frac{\mathbf{W}}{W^2} \frac{dW}{dt}\right)$$
$$= \mathbf{N} \cdot \left(-\mathbf{N} \frac{d\alpha}{dt} + \frac{\mathbf{N} \cdot \mathbf{W}}{W^2} \frac{dW}{dt}\right) \tag{A.15}$$
$$= -\frac{d\alpha}{dt} + 0$$

Now

$$\boldsymbol{\omega} = 0 + \boldsymbol{\omega}_{\mathrm{N}} + \boldsymbol{\omega}_{\mathrm{W}} \tag{A.16}$$

and

$$\left(\frac{d\mathbf{N}}{dt}\right) = \left(\frac{d\mathbf{N}}{dt}\right)_{\mathrm{n}} + 0 + \left(\frac{d\mathbf{N}}{dt}\right)_{\mathrm{W}} \tag{A.17}$$

hence

$$\frac{\boldsymbol{\omega}}{\rho} \cdot \frac{d\mathbf{N}}{dt} = \frac{\boldsymbol{\omega}_{\mathrm{W}}}{\rho} \cdot \left(\frac{d\mathbf{N}}{dt}\right)_{\mathrm{W}} \tag{A.18}$$

but by eq. (8)

$$\frac{d\mathbf{N}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{W}}{W}\right) \times \mathbf{n} + \frac{\mathbf{W}}{W} \times \frac{d\mathbf{n}}{dt}$$
$$= \frac{d}{dt}\left(\frac{\mathbf{W}}{W}\right)_{\mathrm{N}} \times \mathbf{n} + \frac{\mathbf{W}}{W} \times \left(\frac{d\mathbf{n}}{dt}\right)_{\mathrm{N}}$$

by eq. (14)

$$= -\mathbf{N} \times \mathbf{n} \frac{d\alpha}{dt} + \frac{\mathbf{W}}{W} \times \left(\frac{d\mathbf{n}}{dt}\right)_{\mathrm{N}}$$

by eq. (1.10)

$$= +\frac{\mathbf{W}}{W} \frac{d\alpha}{dt} + \left(\frac{\mathbf{W}}{W} \times \left(\frac{d\mathbf{n}}{dt}\right)_{\mathrm{N}}\right)_{\mathrm{n}} \tag{A.19}$$

and by eqs. (18) and (19)

$$\frac{\boldsymbol{\omega}}{\rho} \cdot \frac{d\mathbf{N}}{dt} = \frac{\omega_{\mathrm{W}}}{\rho} \frac{d\alpha}{dt} \tag{A.20}$$

Now

$$\frac{\mathbf{N}}{\rho} \cdot \text{grad } T \times \text{grad } s = \frac{1}{\rho} \text{grad } T \cdot \text{grad } s \times \mathbf{N}$$

by eq. (1.9)

$$= \frac{1}{\rho} \text{grad } T \cdot \mathbf{n} \frac{\partial s}{\partial n} \times \mathbf{N}$$

by eq. (1.8)

$$= \frac{1}{\rho} \text{grad } T \cdot \frac{\mathbf{W}}{W} \frac{\partial s}{\partial n}$$
$$= \frac{1}{W\rho} \frac{\partial s}{\partial n} \frac{dT}{dt} \tag{A.21}$$

but

$$I \equiv C_p T + \frac{W^2 - U^2}{2} \tag{A.22}$$

and by section 1

$$\frac{dI}{dt} = 0$$

hence

$$\frac{dT}{dt} = \frac{-1}{C_p} \frac{d}{dt} \left(\frac{W^2 - U^2}{2}\right) \tag{A.23}$$

Now s is constant on the streamsurface so that the difference in s on two adjacent streamsurfaces separated by the normal distance $\mathbf{t}'$ is constant and may be written

$$\mathbf{t}'_1 \text{ grad}_1\, s = \mathbf{t}' \,.\, \text{grad } s \tag{A.24}$$

where subscript 1 refers to some arbitrary reference point on the streamsurface, say upstream. By eqs. (8) and (24)

$$\mathbf{t}'_1 \left(\frac{\partial s}{\partial n}\right)_1 = \mathbf{t}' \frac{\partial s}{\partial n}$$

hence by eq. (2.6)

$$\frac{\partial s}{\partial n} = \frac{t'_1}{t'}\left(\frac{\partial s}{\partial n}\right)_1 = \frac{C_p}{t'} C_1 \tag{A.25}$$

then by eqs. (16), (18) and (20), (21), (23)

$$\frac{\mathbf{N}}{\rho} \,.\, \text{grad } T \times \text{grad } s = -C_1 \frac{1}{\rho W t'} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W^2 - U^2}{2}\right) \tag{A.26}$$

where C_1 is given at eq. (2.6) so that by eqs. (5), (12), (15) and (26), eq. (4) may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_N}{\rho}\right) = -\frac{\omega_N}{\rho}\frac{1}{\rho W t'}\frac{\mathrm{d}}{\mathrm{d}t}(\rho W t') - \frac{\omega_W}{\rho}\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\omega_W}{\rho}\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{C_1}{\rho W t'}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W^2 - U^2}{2}\right)$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_N W t' + C_1 \frac{W^2 - U^2}{2}\right) = 0 \tag{A.27}$$

When $\mathbf{b} = (\mathbf{W}/W)$ then from eq. (2.3)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho}\right) = \frac{\mathbf{W}}{W}\cdot\left(\frac{\boldsymbol{\omega}}{\rho} \,.\, \text{grad}\right)\mathbf{W} + \frac{\boldsymbol{\omega}}{\rho} \,.\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right) + \frac{1}{\rho}\frac{\mathbf{W}}{W} \,.\, \text{grad } T \times \text{grad } s \tag{A.28}$$

but

$$\frac{\mathbf{W}}{W}\cdot\left(\frac{\boldsymbol{\omega}}{\rho} \,.\, \text{grad}\right)\mathbf{W} = \frac{\boldsymbol{\omega}}{\rho}\cdot\left(\frac{1}{W}\text{grad}\right)\frac{W^2}{2} \tag{A.29}$$

and

$$\frac{1}{W}\text{grad}\frac{W^2}{2} \equiv \frac{1}{W}(\mathbf{W} \,.\, \text{grad } \mathbf{W} - (\text{curl } \mathbf{W}) \times \mathbf{W}) = \frac{1}{W}\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} - \text{curl }(\mathbf{V} - \mathbf{U}) \times \frac{\mathbf{W}}{W}$$

by eq. (13)

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right) + \frac{\mathbf{W}}{W^2}\frac{\mathrm{d}W}{\mathrm{d}t} - (\boldsymbol{\omega} - 2\boldsymbol{\Omega}) \times \frac{\mathbf{W}}{W} \tag{A.30}$$

therefore by eqs. (29) and (30)

$$\frac{\mathbf{W}}{W}\cdot\left(\frac{\boldsymbol{\omega}}{\rho} \,.\, \text{grad}\right)\mathbf{W} = \frac{\boldsymbol{\omega}}{\rho}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right)_N + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right)_n\right) + \frac{\boldsymbol{\omega} \,.\, \mathbf{W}}{\rho W^2}\frac{\mathrm{d}W}{\mathrm{d}t} - \frac{\boldsymbol{\omega}}{\rho} \,.\, \boldsymbol{\omega} \times \frac{\mathbf{W}}{W} + 2\boldsymbol{\Omega} \,.\, \frac{\mathbf{W}}{W} \times \frac{\boldsymbol{\omega}}{\rho}$$

since $\boldsymbol{\omega}_n = 0$

$$= \frac{\omega_N}{\rho} \,.\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right)_N + \frac{\omega_W}{\rho}\frac{1}{W}\frac{\mathrm{d}W}{\mathrm{d}t} - 0 + 2\boldsymbol{\Omega} \,.\, \frac{\mathbf{W}}{W} \times \frac{\boldsymbol{\omega}_N}{\rho}$$

by eq. (14)

$$= -\frac{\omega_N}{\rho}\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\omega_W}{\rho}\frac{1}{W}\frac{\mathrm{d}W}{\mathrm{d}t} - 2\frac{\omega_N}{\rho}\boldsymbol{\Omega} \,.\, \mathbf{n} \tag{A.31}$$

As above in eq. (31)

$$\frac{\boldsymbol{\omega}}{\rho} \,.\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right) = \frac{\boldsymbol{\omega}}{\rho} \,.\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right)_N = -\frac{\omega_N}{\rho}\frac{\mathrm{d}\alpha}{\mathrm{d}t} \tag{A.32}$$

Now

$$\frac{1}{\rho}\frac{\mathbf{W}}{W} \,.\, \text{grad } T \times \text{grad } s = \frac{1}{\rho}\text{grad } T \,.\, \text{grad } s \times \frac{\mathbf{W}}{W}$$

and by eq. (25)

$$= \frac{1}{\rho}\text{grad } T \,.\, \frac{t'_1}{t'}\left(\frac{\partial s}{\partial n}\right)_1 \mathbf{n} \times \frac{\mathbf{W}}{W}$$

by eq. (8)

$$= -\frac{1}{\rho}\frac{t'_1}{t'}\left(\frac{\partial s}{\partial n}\right)_1 (\mathbf{N} \,.\, \text{grad } T) \tag{A.33}$$

but by eq. (22) since $\mathbf{N} \,.\, \text{grad } I = 0$ then

$$\mathbf{N} \,.\, \text{grad } T = \frac{1}{C_p}\left\{-\mathbf{N} \,.\, \text{grad}\left(\frac{W^2 - U^2}{2}\right)\right\} \tag{A.34}$$

so that by eqs. (33), (34) and (16)

$$\frac{1}{\rho}\frac{\mathbf{W}}{W} \,.\, \text{grad } T \times \text{grad } s = C_1 \frac{1}{\rho t'}\left(W\frac{\partial W}{\partial N} - \Omega^2 r \mathbf{N} \,.\, \text{grad } r\right) \tag{A.35}$$

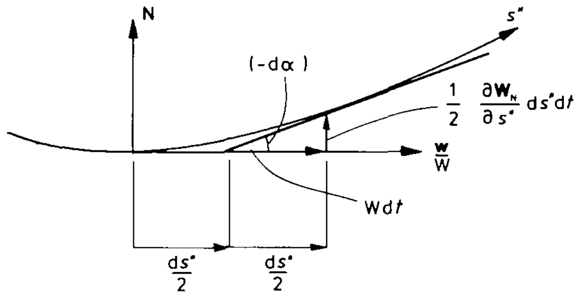
but

$$\boldsymbol{\omega} \equiv \text{curl } \mathbf{V} = \text{curl } \mathbf{W} + 2\boldsymbol{\Omega} \tag{A.36}$$

while

$$(\text{curl } \mathbf{W})_n = \frac{\partial W}{\partial N} - \frac{\partial W_N}{\partial s''} \tag{A.37}$$

Now by geometry to the sign convention of eq. (14)



$$-\mathrm{d}\alpha = \frac{1}{2}\frac{\partial W_N}{\partial s''}\frac{\mathrm{d}s''\,\mathrm{d}t}{\mathrm{d}s''/2}$$

$$\frac{\partial W_N}{\partial s''} = -\frac{\mathrm{d}\alpha}{\mathrm{d}t} \tag{A.38}$$

but since $\omega_n = 0$ then by eqs. (36), (37) and (38)

$$0 = \frac{\partial W}{\partial N} + \frac{\mathrm{d}\alpha}{\mathrm{d}t} + 2\mathbf{\Omega}\,.\,\mathbf{n}$$

hence

$$\frac{\partial W}{\partial N} = -\frac{\mathrm{d}\alpha}{\mathrm{d}t} - 2\mathbf{\Omega}\,.\,\mathbf{n} \tag{A.39}$$

Since

$$\text{grad } r = \frac{\mathbf{r}}{r} \tag{A.40}$$

then by eqs. (35), (39) and (40)

$$\frac{1}{\rho}\frac{\mathbf{W}}{W}\,.\,\text{grad } T \times \text{grad } s = \frac{C_1}{\rho t'} \times \left(-W\frac{\mathrm{d}\alpha}{\mathrm{d}t} - 2W\mathbf{\Omega}\,.\,\mathbf{n} - \Omega^2\mathbf{N}\,.\,\mathbf{r}\right) \tag{A.41}$$

Now by eqs. (31), (32) and (41), eq. (28) may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho}\right) = -2\frac{\omega_N}{\rho}\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\omega_W}{\rho}\frac{1}{W}\frac{\mathrm{d}W}{\mathrm{d}t} - 2\frac{\omega_N}{\rho}\mathbf{\Omega}\,.\,\mathbf{n},$$
$$-\frac{C_1}{\rho t'}W\frac{\mathrm{d}\alpha}{\mathrm{d}t}\,.\, -\frac{C_1}{\rho t'}(2W\mathbf{\Omega}\,.\,\mathbf{n} + \Omega^2\mathbf{N}\,.\,\mathbf{r})$$

then by rearrangement and collecting terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho W}\right) = -\left(2\frac{\omega_N}{\rho W} + \frac{C_1}{\rho t'}\right)\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \left(2\frac{\omega_N}{\rho W} + \frac{C_1}{\rho t'}\right)\mathbf{\Omega}\,.\,\mathbf{n}$$
$$-\frac{C_1}{\rho t'}\left(\mathbf{\Omega}\,.\,\mathbf{n} + \frac{\Omega^2\mathbf{N}\,.\,\mathbf{r}}{W}\right)$$

and substituting from eq. (15), C_2 from eq. (2.7a)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_W}{\rho W}\right) = -\frac{(2C_2 + C_1U^2)}{\rho W^2 t'}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \mathbf{\Omega}\,.\,\mathbf{n}\right)$$
$$-\frac{C_1}{\rho t'}\left(\mathbf{\Omega}\,.\,\mathbf{n} + \frac{\Omega^2\mathbf{N}\,.\,\mathbf{r}}{W}\right) \tag{A.42}$$

Consider a closed contour on the Bernoulli surface (Fig. 3) consisting of elements dl and

$$\mathrm{d}l + \frac{\partial(\mathrm{d}l)}{\partial N}\mathrm{d}N$$

of two adjacent streamlines and elements dN and

$$\mathrm{d}N + \frac{\partial(\mathrm{d}N)}{\partial l}\mathrm{d}l$$

of two adjacent normals, then around this contour

$$\oint\frac{\mathbf{W}\,.\,\mathrm{d}\mathbf{l}}{W^2}$$
$$= \left\{\left(\frac{\mathrm{d}l}{W}\right) + \frac{\partial}{\partial N}\left(\frac{\mathrm{d}l}{W}\right)\mathrm{d}N\right\} - \left(\frac{\mathrm{d}l}{W}\right) = \frac{\partial}{\partial N}\left(\frac{\mathrm{d}l}{W}\right)\mathrm{d}N \tag{A.43}$$

because there is no contribution to **W** . d**l** along the normals. But

$$\frac{\mathrm{d}\mathbf{l}}{W} = \mathrm{d}t \tag{A.44}$$

is the time taken for a particle to traverse the distance dl. Equation (43) then expresses the difference in time for particles on adjacent streamlines to traverse between two adjacent normals. But by definition

$$\oint\frac{\mathbf{W}\,.\,\mathrm{d}\mathbf{l}}{W^2} = \text{curl}\left(\frac{\mathbf{W}}{W^2}\right).\,\mathrm{d}\mathbf{S} \tag{A.45}$$

where

$$\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S = \mathrm{d}\mathbf{N} \times \mathrm{d}\mathbf{l} = \mathbf{n}\,\mathrm{d}N\,\mathrm{d}l \tag{A.46}$$

is the surface element enclosed by the contour. Now,

$$\text{curl}\left(\frac{\mathbf{W}}{W^2}\right).\,\mathrm{d}\mathbf{S} = \left\{\frac{1}{W^2}\text{curl }\mathbf{W} + \text{grad}\left(\frac{1}{W^2}\right) \times \mathbf{W}\right\}.\,\mathrm{d}\mathbf{S}$$
$$= \frac{1}{W^2}(\boldsymbol{\omega} - 2\mathbf{\Omega})\,.\,\mathbf{n}\,\mathrm{d}S$$
$$-\frac{2}{W^2}\text{grad } W\,.\,\frac{\mathbf{W}}{W} \times \mathrm{d}\mathbf{S}$$

since $\omega_n = 0$

$$= -\frac{2\mathbf{\Omega}\,.\,\mathbf{n}}{W^2}\mathrm{d}S - \frac{2}{W^2}$$
$$\times \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{W}}{W}\right) - \frac{\mathbf{W}}{W^2}\frac{\mathrm{d}W}{\mathrm{d}t} - \boldsymbol{\omega}' \times \frac{\mathbf{W}}{W}\right\}.\,\mathbf{N}\,\mathrm{d}S$$
$$= -\frac{2\mathbf{\Omega}\,.\,\mathbf{n}}{W^2}\mathrm{d}S - \frac{2}{W^2}$$
$$\times \left(-\frac{\mathrm{d}\alpha}{\mathrm{d}t} - 0 - 0 + 2\mathbf{\Omega} \times \frac{\mathbf{W}}{W}\,.\,\mathbf{N}\right)\mathrm{d}S$$
$$= \frac{2}{W^2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \mathbf{\Omega}\,.\,\mathbf{n}\right)\mathrm{d}S \tag{A.47}$$

hence by eqs. (43), (44), (45), (46) and (47)

$$\frac{1}{dt}\frac{\partial(dt)}{\partial N} = \frac{2}{W}\left(\frac{d\alpha}{dt} + \mathbf{\Omega} \, . \, \mathbf{n}\right) \qquad (A.48)$$

Let the mass flow rate through the elemental streamtube described by this contour be

$$d\dot{m} = \rho W t' \, dN \qquad (A.49)$$

then

$$\frac{1}{d\dot{m}}\frac{\partial(dt)}{\partial N}\frac{dN}{dt} = \frac{2}{\rho W^2 t'}\left(\frac{d\alpha}{dt} + \mathbf{\Omega} \, . \, \mathbf{n}\right) \qquad (A.50)$$

which substituted in eq. (42) yields

$$\frac{d}{dt}\left(\frac{\omega_W}{\rho W}\right) = -\frac{2C_2 + C_1 U^2}{d\dot{m}}\frac{\partial(dt)}{\partial N}\frac{dN}{dt} - \frac{C_1}{\rho t'}\left(\mathbf{\Omega} \, . \, \mathbf{n} + \frac{\Omega^2}{W}\mathbf{N} \, . \, \mathbf{r}\right) \qquad (A.51)$$